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Approximate and Exact Consistency of Histories.

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Abstract

The consistent histories formalism is discussed using path-projected states. These are used to analyse various criteria for approximate consistency. The connection between the Dowker-Halliwell criterion and sphere packing problems is shown and used to prove several new bounds on the violation of probability sum rules. The quantum Zeno effect is also analysed within the consistent histories formalism and used to demonstrate some of the difficulties involved in discussing approximate consistency. The complications associated with null histories and infinite sets are briefly discussed.

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I. INTRODUCTION

The Copenhagen interpretation of Quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own interpretation [1,2]. This problem is particularly acute in quantum cosmology, since it is highly unlikely that any systems obeying classical mechanics existed in the early universe.

The consistent histories approach to quantum mechanics is an attempt to remove the ambiguities and difficulties inherent in the Copenhagen interpretation. The basic objects are sequences of events or *histories*. A set of histories must include all possibilities and must be *consistent*. The individual histories can then be considered physical possibilities with definite probabilities, and they obey the ordinary rules of probability and logical inference. The predictions of the consistent histories formalism are identical to the predictions of standard quantum mechanics where laboratory experiments are concerned, but they take place within a more general theory.

Much work has been done on trying to understand the emergence of classical phenomena within the consistent histories approach [3–14]. These studies consider closed quantum systems in which the degrees of freedom are split between an unobserved environment and distinguished degrees of freedom such as the position of the centre of mass.

In these and other realistic models it is often hard to find physically interesting, *exactly* consistent sets, so most examples studied are only approximately consistent. These models do show, however, that histories consisting of projections onto the distinguished degrees of freedom at discrete times are *approximately* consistent. This work is necessary for explaining the emergence of classical phenomena but is incomplete. The implications of different definitions differences of *approximate* consistency have received little research: the subject is more complicated than has sometimes been realised. A quantitative analysis of the quantum Zeno paradox demonstrates some of the problems. A deeper problem is explaining why quasi-classical sets of histories occur as opposed to any of the infinite number of consistent,

non-classical sets. Until these problems are understood the program is incomplete.

In this paper I examine two different criteria approaches to approximate consistency and analyse two frequently used criteria. I show a simple relation with sphere-packing problems and use this to provide a new bound on probability changes under coarse-grainings.

A. Consistent Histories Formalism

The most basic objects in the consistent histories formulation are projection operators, representing particular states of affairs existing at particular times [15]. These are combined into time-ordered strings which are the elementary events, or *histories*, in the probability sample space S. A set of projective decompositions of the identity $\{\sigma_n, \ldots, \sigma_1\}$ and times $\{t_n, \ldots, t_1\}$ define a set of *class-operators* $S = \{C_\alpha\}$,

$$\begin{aligned} C_\alpha = U(-t_n)P^n_{\alpha_n}U(t_n - t_{n-1})P^{n-1}_{\alpha_{n-1}}\cdots \\ U(t_2 - t_1)P^1_{\alpha_1}U(t_1), \end{aligned} \tag{1.1}$$

where $P^j_{\alpha_j} \in \sigma_j$ and $U(t)$ is the time evolution operator. The explicit time dependence of this set can be removed by defining new sets of projectors $\sigma'_k = U(-t_k)\sigma^k_{\alpha_k}U(t_k)$.

More general sets of class-operators can be created by *coarse-graining.* $S^* = \{C^*_\beta\}$ is a coarse-graining of S if $C^*_\beta = \sum_{\alpha\in\overline{\alpha}_\beta} C_\alpha$, where $\{\overline{\alpha}_\beta\}$ is a partition of S. Omnès defines sets of histories without any coarse-graining as Type I, and those which have been coarse-grained but where the class-operators are still strings of projectors as Type II [16]. I shall follow Isham [17] and call these *homogenous.*

Gell-Mann and Hartle consider completely general coarse-grainings, and also ones which they call branch-dependent [3]. These are a restriction of Type II histories to those in which earlier projections are independent of later ones.

The type of histories used makes very little difference to discussions of approximate consistency. I will therefore follow Gell-Mann and Hartle and use completely general class-operators, though on occasion I shall state stronger results which hold for homogenous class-operators.

Probabilities are defined by the formula

$$P(\alpha) = D_{\alpha\alpha}, \tag{1.2}$$

where $D_{\alpha\beta}$ is the decoherence matrix

$$D_{\alpha\beta} = \mathrm{Tr}(C_\alpha \rho C_\beta{}^\dagger), \tag{1.3}$$

and where ρ is the initial density matrix[1]. If no further conditions were imposed these probabilities could contradict ordinary quantum mechanics: they would be inconsistent. A necessary and sufficient requirement for consistency is that the probability of a collection of histories should not change under *any* coarse-graining [18][2]. This condition can be expressed

$$\mathrm{Re}(D_{\alpha\beta}) = 0, \quad \forall \alpha \neq \beta, \tag{1.4}$$

which Gell-Mann and Hartle call *weak consistency*. A stronger condition,

$$D_{\alpha\beta} = 0, \quad \forall \alpha \neq \beta, \tag{1.5}$$

is often used in the literature for simplicity[3]. I shall restrict my discussion to the weak condition (1.4), as it is necessary physically and any consequent results will certainly hold if a stronger condition is satisfied.

B. Path-Projected States

A simple way of regarding a set of histories is as a set of path-projected states or *history states*[4]. For a pure initial density matrix $\rho = |\psi\rangle\langle\psi|$ these states are defined by

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[1]Generalisations exist that also have a final density matrix [15].

[2]Griffiths and Omnès only consider more restricted sets of histories, which results in a weaker condition [15,16].

[3]Gell-Mann and Hartle call this medium consistency, and also define two even stronger conditions [3].

[4]This approach loses its advantages if a final density matrix is present.

$$\mathbf{u}_\alpha = C_\alpha|\psi\rangle \in \mathbf{H_1}, \tag{1.6}$$

where $\dim(\mathbf{H_1}) = d$. For a mixed density matrix,

$$\rho = \sum_{i=1}^{n} p_i|\psi_i\rangle_1\langle\psi_i|_1 \qquad |\psi_i\rangle_1 \in \mathbf{H}_1, \tag{1.7}$$

history states can be defined by regarding ρ as a reduced density matrix of a pure state in a larger Hilbert space $\mathbf{H}_1 \otimes \mathbf{H}_2$, where $\mathbf{H}_2$ is of dimension $\mathrm{rank}(\rho) = n$ (possibly infinite), with orthonormal basis $|i\rangle_2$. All operators A_1 on $\mathbf{H}_1$, can be extended to operators on $\mathbf{H}_1 \otimes \mathbf{H}_2$ by defining $A = A_1 \otimes \mathbf{1}_2$. The state in the larger space is

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}|\psi_i\rangle_1 \otimes |i\rangle_2 \qquad |\psi\rangle \in \mathbf{H}_1 \otimes \mathbf{H}_2, \tag{1.8}$$

and the history states are again given by equation (1.6); but now they are vectors in an $N = nd$ dimensional Hilbert space.

The decoherence matrix (1.3) is

$$D_{\alpha\beta} = \mathrm{Tr}(\mathbf{u}_\alpha \mathbf{u}_\beta^\dagger) = \mathbf{u}_\beta^\dagger \mathbf{u}_\alpha, \tag{1.9}$$

so the probability of the history α occurring is $||\mathbf{u}_\alpha||^2$. The consistency equations (1.4) are

$$\mathrm{Re}({\mathbf{u}_\alpha}^\dagger \mathbf{u}_\beta) = 0 \quad \forall\, \alpha \neq \beta. \tag{1.10}$$

A complex Hilbert space of dimension N is isomorphic to the real Euclidean space $\mathbf{R}^{2N}$. The consistency condition (1.10) takes on an even simpler form when the history states are regarded as vectors in the real Hilbert space. I define the *real history states*

$$\mathbf{v}_\alpha = \mathrm{Re}(\mathbf{u}_\alpha) \oplus \mathrm{Im}(\mathbf{u}_\alpha) \quad \in \mathbf{R}^{2N}, \tag{1.11}$$

and then the consistency condition (1.10) is that the set of real history states, $\{\mathbf{v}_\alpha\}$, is orthogonal,

$$\mathbf{v}_\alpha^T \mathbf{v}_\beta = 0 \quad \forall\, \alpha \neq \beta. \tag{1.12}$$

The probabilities of history α is $||\mathbf{v}_\alpha||^2$.

For the rest of this paper I shall only consider pure initial states since the results can easily be extended to the mixed case by using the above methods.

II. APPROXIMATE CONSISTENCY

In realistic examples it is often difficult to find physically interesting, *exactly* consistent sets. This rarity impacts upon the use of consistent histories in studies of dust particles or oscillators coupled to environments [3–14]. Frequently in these studies, the off-diagonal terms in the decoherence matrix decay exponentially with the time between projections, but their real parts are never exactly zero, so the histories are only approximately consistent. Therefore if the histories are coarse-grained, the probabilities for macroscopic events will vary very slightly depending on the exact choice of histories in the set. Because the probabilities can be measured experimentally, they should be unambiguously predicted — at least to within experimental precision.

In his seminal work Griffiths states that "violations of [the consistency criterion (1.4)] should be so small that physical interpretations based on the weights [probabilities] remain essentially unchanged if the latter are shifted by amounts comparable with the former" [15, sec. 6.2]. Omnès [16,19–24], and Gell-Mann and Hartle [3,18,25] make the same point. The amount by which the probabilities change under coarse-graining is the extent to which they are ambiguous. I shall define the the largest such change in a set to be the *maximum probability violation* or MPV.

Dowker and Kent [26,27] argue that more is needed. Why should *approximately* consistent sets be used? They suggest that "near" a generic approximately consistent set there will be an exactly consistent one. "Near" means that the two sets describe the same physical events to order ϵ; the relative probabilities and the projectors must be the same to order ϵ. In the paper I investigate which criteria will guarantee this, and show that some of the commonly used ones are not sufficient.

A. Probability Violation

The MPV can be defined equivalently in terms of the decoherence matrix:

$$
\begin{aligned}
\mathrm{MPV}(D) &= \max_{\overline{\alpha}} \left| P(\overline{\alpha}) - \sum_{\alpha\in\overline{\alpha}} P(\alpha) \right|, &(2.1)\\
&= \max_{\overline{\alpha}} \left| \sum_{\alpha,\beta\in\overline{\alpha}} D_{\alpha\beta} - \sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha} \right|, &(2.2)\\
&= \max_{\overline{\alpha}} \left| \sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta} \right|. &(2.3)
\end{aligned}
$$

The maximum is taken over all possible coarse-grainings $\overline{\alpha}$. For large sets of histories this is difficult to calculate as the number of possible coarse-grainings is $O(2^n)$. A simple criterion that if satisfied to some order $\epsilon(\delta)$ would ensure that the MPV were less than δ would be preferable here.

This is not a trivial problem. The frequently used criterion [3,23]

$$
|D_{\alpha\beta}| \leq \epsilon(\delta), \quad \forall\alpha\neq\beta \tag{2.4}
$$

is not sufficient for any $\epsilon(\delta) > 0$. Theorem (E) shows that for any $\epsilon(\delta) > 0$ there are finite sets of histories satisfying (2.4) with an arbitrarily large MPV. The example used in the proof also shows some of the complications that arise when discussing infinite sets of histories. All sets of histories in the rest of the paper will be assumed to be finite unless otherwise stated.

A simple bound[5] for the MPV is

$$
\mathrm{MPV}(D) \leq \sum_{\alpha\neq\beta} |\mathrm{Re}(D_{\alpha\beta})|. \tag{2.5}
$$

This leads to the criterion for the individual elements

$$
|\mathrm{Re}(D_{\alpha\beta})| \leq \frac{\delta}{n(n-1)}, \quad \forall\alpha\neq\beta, \tag{2.6}
$$

where n is the number of histories. Equation (2.6) ensures that the MPV is less than δ, although the condition will generally be much stronger than necessary. It would be

[5] When the class-operators are homogenous the bound can be improved to $M(D_{\alpha\beta}) \leq 1/2\sum_{\alpha\neq\beta}|\mathrm{Re}(D_{\alpha\beta})|$, since $\sum_{\alpha\neq\beta} D_{\alpha\beta} = 0$.

preferable however, to have a criterion that only depended on the Hilbert space and not on the particular set of histories.

III. THE DOWKER-HALLIWELL CRITERION

Dowker and Halliwell discussed approximate consistency in their paper [6], in which they introduced a new criterion[6]

$$|\mathrm{Re}(D_{\alpha\beta})| \le \epsilon\,(D_{\alpha\alpha}D_{\beta\beta})^{1/2}, \quad \forall\,\alpha \ne \beta, \tag{3.1}$$

which I shall call the *Dowker-Halliwell criterion* or DHC. Using the central limit theorem and assuming that the off-diagonal elements are independently distributed, Dowker and Halliwell demonstrate that (3.1) implies

$$\left|\sum_{\alpha\ne\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \le \epsilon \sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha}, \tag{3.2}$$

for most coarse-grainings $\overline{\alpha}$. This is a natural generalisation of (1.4) to saying that the probability sum rules are satisfied to *relative* order ϵ. For homogenous histories this is a similar but stronger condition than requiring that the MPV (2.1) is less than ϵ, since

$$\sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha} \le \sum D_{\alpha\alpha} = 1. \tag{3.3}$$

But for general class-operators $\sum D_{\alpha\alpha}$ is unbounded and (3.2) must either be modified or supplemented by a condition such as

$$\left|\sum D_{\alpha\alpha} - 1\right| \le \epsilon. \tag{3.4}$$

This is only a very small change and for approximately consistent sets is not significant.

For the sake of completeness, I shall occasionally mention a similar criterion which I shall call the *medium DHC*

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[6] I have replaced Dowker-Halliwell's $<$ with $\le$ to avoid problems with histories of zero probability.

$$|D_{\alpha\beta}| \leq \epsilon \, (D_{\alpha\alpha} D_{\beta\beta})^{1/2}, \quad \forall \, \alpha \neq \beta. \tag{3.5}$$

As Dowker and Halliwell point out, [28], the off-diagonal terms are often not well modelled as independent random variables. Indeed even when this assumption is valid, the MPV will usually be much higher. By appropriately choosing ϵ as a function of δ, however, it is possible to eliminate these problems, and to utilise the many other useful properties of the DHC.

A. Geometrical Properties

The Dowker-Halliwell criterion has a simple geometrical interpretation. In terms of the real history states (1.11) the DHC can be written (ignoring null histories[7])

$$\frac{|\mathbf{v}_\alpha^T \mathbf{v}_\beta|}{||\mathbf{v}_\alpha|| \, ||\mathbf{v}_\beta||} = |\cos(\theta_{\alpha\beta})| \leq \epsilon \quad \forall \, \alpha \neq \beta, \tag{3.6}$$

where $\theta_{\alpha\beta}$ is the angle between the real history vectors $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$. The DHC requires that the angle between every pair of histories must be at least $\cos^{-1}\epsilon$ degrees.

In a d dimensional Hilbert space there can only be $2d$ exactly consistent, non-null histories. Thus, if a set contains more than $2d$ non-null histories, it cannot be continuously related to an exactly consistent set unless some of the histories become null. Establishing the maximum number of histories satisfying (3.6) in finite dimensional spaces is a particular case from a family of problems, which has received considerable study.

B. Generalised Kissing Problem

The Generalised Kissing Problem is the problem of determining how many $(k-1)$-spheres of radius r can be placed on the surface of a sphere with radius R in $\mathbf{R}^k$. This problem is equivalent to calculating the maximum number of points that can be found on the sphere all at least $\cos^{-1}\epsilon$ degrees apart, where $\epsilon = 1 - 2r^2(R+r)^{-2}$.

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[7]A null history is one with probability equal to 0.

To express these ideas mathematically, I define $M(\mathbf{L}, (\mathbf{u}, \mathbf{v}) \leq s)$ to be the size of the largest subset of $\mathbf{L}$, such that $(\mathbf{u}, \mathbf{v}) \leq s$ for all different elements in the subset, where $\mathbf{L}$ is a metric space. The Generalised Kissing Problem is calculating

$$M(\mathbf{S}^{k-1}, \mathbf{u}^T\mathbf{v} \leq \epsilon), \tag{3.7}$$

where $\mathbf{S}^{k-1}$ is the set of points on the unit sphere in $\mathbf{R}^k$. The greatest number of histories satisfying the DHC is

$$M(\mathbf{CS}^{d-1}, |\mathrm{Re}(\mathbf{u}^\dagger\mathbf{v})| \leq \epsilon) \; = \; M(\mathbf{S}^{2d-1}, |\mathbf{u}^T\mathbf{v}| \leq \epsilon) \tag{3.8}$$

and for the medium DHC is

$$M(\mathbf{CS}^{d-1}, |\mathbf{u}^\dagger\mathbf{v}| \leq \epsilon), \tag{3.9}$$

where $\mathbf{CS}^{d-1}$ is the set of points on the unit sphere in $\mathbf{C}^d$.

There is a large literature devoted to sphere-packing. Although few exact results emerge from this work, numerous methods exist for generating bounds. The tightest upper bounds derive from an optimisation problem. In appendix (A) I prove that the well known bound

$$M(\mathbf{S}^{2d-1}, |\mathbf{u}^T\mathbf{v}| \leq \epsilon) \leq \left\lfloor \frac{2d(1-\epsilon^2)}{1-2d\epsilon^2} \right\rfloor \tag{3.10}$$

is the solution to the optimisation problem when $\epsilon^2 \leq 1/(2d+2)$.

The most important feature of this bound is that for $0 \leq \epsilon \leq 1/(2d)$ it is exact, since for $\epsilon < 1/(2d)$ it gives $2d$ as an upper bound and for $\epsilon = 1/(2d)$ it gives $2d+1$, and there are packings that achieve these bounds[8]. This is also the range of most interest in Consistent Histories since an exactly consistent set cannot contain more than $2d$ non-null histories. This result shows that if $\epsilon < 1/(2d)$ then there cannot be more than $2d$ histories in a set satisfying the DHC. Deciding when a set of vectors could be a set of histories is a difficult problem, so this result does not prove that this bound is optimal, although it is suggestive.

[8]A packing with $\epsilon = 1/(2d)$ is generated by the rays passing through the $2d+1$ vertices of the regular $(2d)$-simplex.

This bound (3.10) can now be used to prove several upper bounds on probability sum rules.

$$\left|\sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \leq \sum_{\alpha\neq\beta\in\overline{\alpha}} |D_{\alpha\beta}|, \tag{3.11}$$

$$\leq \epsilon \sum_{\alpha\neq\beta\in\overline{\alpha}} \left(D_{\alpha\alpha}D_{\beta\beta}\right)^{1/2}, \tag{3.12}$$

$$\leq \epsilon(n-1)\sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha}. \tag{3.13}$$

But the number of histories n is bounded by $2d(1-\epsilon^2)/(1-2d\epsilon^2)$, so

$$\left|\sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \leq \epsilon\,\frac{2d-1}{1-2d\epsilon^2}\sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha}. \tag{3.14}$$

Choose

$$\epsilon(\delta) = \frac{-(2d-1)+\sqrt{(2d-1)^2+8d\delta^2}}{4d\delta} \tag{3.15}$$

and then (3.14) implies

$$\left|\sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \leq \delta\sum_{\alpha\in\overline{\alpha}} D_{\alpha\alpha}. \tag{3.16}$$

This is the exact version of Dowker and Halliwell's result (3.2). For homogenous histories $\sum_\alpha D_{\alpha\alpha}=1$ and then (3.14) and (3.15) imply

$$\mathrm{MPV} < \delta. \tag{3.17}$$

These results can easily be extended to general class-operators since the same methods lead to a bound on $\sum_\alpha D_{\alpha\alpha}$ in terms of ϵ.

$$\sum_{\alpha,\beta} D_{\alpha\beta} = 1 \tag{3.18}$$

$$\Rightarrow \sum_{\alpha} D_{\alpha\alpha} = 1 - \sum_{\alpha\neq\beta} D_{\alpha\beta} \tag{3.19}$$

$$\Rightarrow \sum_{\alpha} D_{\alpha\alpha} \leq 1 + \sum_{\alpha\neq\beta} |D_{\alpha\beta}| \tag{3.20}$$

$$\leq 1 + \epsilon(n-1)\sum_{\alpha} D_{\alpha\alpha} \tag{3.21}$$

$$\Rightarrow \sum_{\alpha} D_{\alpha\alpha} \leq \frac{1}{1-(n-1)\epsilon}. \tag{3.22}$$

There are sets of histories for which this bound is obtained. In particular if $\epsilon = 1/(n-1)$ there are finite sets for which $\sum_\alpha D_{\alpha\alpha}$ is arbitrarily large. Inserting this result into (3.14) results in

$$\left|\sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \leq \frac{\epsilon(n-1)}{1-(n-1)\epsilon} \tag{3.23}$$

$$\leq \epsilon\frac{2d-1}{1+\epsilon-2d\epsilon(1+\epsilon)}. \tag{3.24}$$

Choose

$$\epsilon(\delta) = \frac{-(2d-1)(1+\delta)+\sqrt{(2d-1)^2(1+\delta)^2+8d\delta^2}}{4d\delta}, \tag{3.25}$$

and then (3.24) becomes

$$\left|\sum_{\alpha\neq\beta\in\overline{\alpha}} D_{\alpha\beta}\right| \leq \delta, \tag{3.26}$$

so

$$MPV \leq \delta. \tag{3.27}$$

For physical situation the probability violation must be small so $\delta << 1$ and these results can be simplified. From (3.10) if $\epsilon < 1/(2d)$ $n \leq 2d$ so (3.15) and (3.25) can be simplified to

$$\epsilon(\delta) = \frac{\delta}{2d}, \delta < 1 \Rightarrow MPV \leq \delta + O(\delta^2), \tag{3.28}$$

for all types of histories. This is the main result of this paper. If the medium DHC holds *or* the class-operators are homogenous then (3.28) can be weakened to $\epsilon(\delta) = \delta/d$ and still imply (3.17). If the medium DHC holds *and* the class-operators are homogenous then (3.28) can be further weakened to $\epsilon(\delta) = 2\delta/d$ and still imply (3.17).

In appendix (D) I give a simple example of a family of sets of histories, of any size, satisfying the medium DHC with MPV $= d\epsilon/4$. If ϵ is chosen according to (3.28) then the MPV $= \delta/8$. This example illustrates that equation (3.28) is close to the optimal bound. Since the example satisfies the *medium* DHC and the class-operators are homogenous ϵ can

be chosen to be $2\delta/d$ and the MPV is then $\delta/2$, so for this example the bound is achieved within a factor of two.

The choice $\epsilon = \delta/(2d)$ in relation to the DHC is particularly convenient in computer models. Often one constructs a set of histories by individually making projections, and one desires a simple criterion which will bound the MPV. The DHC solves this problem.

The only known lower bounds for the generalised kissing problem derive from an argument of Shannon's [29] developed by Wyner [30]. Shannon proved

$$M(\mathbf{S}^{2d-1},\, |\mathbf{u}^T\mathbf{v}| \le \epsilon) \ge (1-\epsilon^2)^{1/2-d}. \tag{3.29}$$

I explain the proof and extend it for the medium DHC in appendix (A 2).

This simple bound (3.29) has an important consequence: the number of histories satisfying the DHC can increase exponentially with d if ϵ is constant. So for constant $\epsilon > 0$ by choosing a large enough Hilbert space the MPV can be arbitrarily large, therefore ϵ must be chosen according to the dimension of the Hilbert space.

When the Hilbert space is infinite-dimensional and separable, and $\epsilon > 0$, (3.29) suggests that there can be an uncountable number of histories satisfying the DHC. If so, the DHC can only guarantee proximity to an exactly consistent set for finite Hilbert spaces. Though if the system is set up in a Hilbert space of dimension d and the limit $d \to \infty$, $\epsilon = O(\sqrt{\frac{\log d}{d}})$ is taken (assuming it exists) then the bound remains countable and it may be useful even for infinite spaces.

If there are n histories satisfying the DHC with $\epsilon = \delta/(n-1)$, then, from (3.13), MPV $\le$ $\delta + O(\delta^2)$. This result is trivial, but the DHC also ensures that the histories will span a subspace of dimension at least $n/2$. Therefore, there will be exactly consistent sets with the same number of non-null histories that span the same subspace.

If

$$\epsilon \le \left[1-(2d)^{2/(1-2d)}\right]^{1/2} = \left[\frac{2\ln 2d}{2d-1}\right]^{1/2} + O\left\{\left[\frac{\ln d}{d}\right]^{3/2}\right\}$$

then the lower bound is less than the trivial lower bound $M \geq d$. Since the upper bounds (3.10) holds only for $\epsilon \leq O(1/\sqrt(d))$ the two sets of bounds are not mutually useful. The Shannon bound is too poor for small ϵ because it ignores the overlap between spherical caps. A more rigorous bound would add points one by one on the edge of existing caps, and allow for the overlap between them. Unfortunately, there are no useful results in this direction.

C. Discontinuity of the Dowker-Halliwell Criterion

A consistent set of histories can be extended to another consistent set by repeating projector sets at adjacent times. Moreover the two sets of histories are physically equivalent descriptions. Both the preceding statements are true because the class-operators for the two sets are identical, since $P_i P_j = \delta_{ij} P_i$. If a slightly perturbed set of projectors is used the result is more complicated.

For simplicity I shall consider a simple example consisting of a set of n history states $\{\mathbf{u}_\alpha\}$, and a projector P and its complement $\overline{P}$. This is sufficiently general to deal with all cases. Suppose that the set of histories is consistent when extended by $\{P, \overline{P}\}$, so

$$\mathrm{Re}(\mathbf{u}_\alpha^\dagger P \mathbf{u}_\beta) = 0, \tag{3.30}$$

$$\mathrm{Re}(\mathbf{u}_\alpha^\dagger \overline{P} \mathbf{u}_\beta) = 0. \tag{3.31}$$

Then the set extended again by $\{P, \overline{P}\}$ will trivially be consistent, $2n$ of the histories will be unaltered, the other $2n$ will be null. Now suppose the projectors are slightly perturbed without altering their ranks. The perturbed projectors will be related to the old ones by a unitary transformation $P' = U(\epsilon)^\dagger P U(\epsilon)$, and $U(\epsilon)$ can be written $U(\epsilon) = \exp{(i\epsilon A)}$, where A is Hermitian. The set of histories $\{P'P\mathbf{u}_\alpha, P'\overline{P}\mathbf{u}_\alpha, \overline{P'}P\mathbf{u}_\alpha, \overline{P'}\overline{P}\mathbf{u}_\alpha\}$ will no longer necessarily be consistent, but it is approximately consistent to $O(\epsilon)$ since it has only been perturbed to this order from an exactly consistent set. To second order in ϵ

$$PP'P = P - \epsilon^2 PA\overline{P}AP, \tag{3.32}$$

$$\overline{P}P'P = -i\epsilon\overline{P}AP + \epsilon^2\overline{P}A(P - 1/2)AP, \tag{3.33}$$

$$\overline{P}P'\overline{P} = \epsilon^2 \overline{P}APA\overline{P}. \tag{3.34}$$

Using (3.30) and (3.32) the real part of the new decoherence matrix to leading order in ϵ contains blocks like

$$\mathrm{Re}\begin{pmatrix} \mathbf{u}_\alpha^\dagger P\mathbf{u}_\alpha & i\epsilon\mathbf{u}_\alpha^\dagger PA\overline{P}\mathbf{u}_\alpha & -\epsilon^2\mathbf{u}_\alpha^\dagger PA\overline{P}AP\mathbf{u}_\beta & i\epsilon\mathbf{u}_\alpha^\dagger PA\overline{P}\mathbf{u}_\beta \\ -i\epsilon\mathbf{u}_\alpha^\dagger \overline{P}AP\mathbf{u}_\alpha & \epsilon^2\mathbf{u}_\alpha^\dagger \overline{P}APA\overline{P}\mathbf{u}_\alpha & -i\epsilon\mathbf{u}_\alpha^\dagger \overline{P}AP\mathbf{u}_\beta & \epsilon^2\mathbf{u}_\alpha^\dagger \overline{P}APA\overline{P}\mathbf{u}_\beta \\ -\epsilon^2\mathbf{u}_\beta^\dagger PA\overline{P}AP\mathbf{u}_\alpha & i\epsilon\mathbf{u}_\beta^\dagger PA\overline{P}\mathbf{u}_\alpha & \mathbf{u}_\beta^\dagger P\mathbf{u}_\beta & i\epsilon\mathbf{u}_\beta^\dagger PA\overline{P}\mathbf{u}_\beta \\ -i\epsilon\mathbf{u}_\beta^\dagger \overline{P}AP\mathbf{u}_\alpha & \epsilon^2\mathbf{u}_\beta^\dagger \overline{P}APA\overline{P}\mathbf{u}_\alpha & -i\epsilon\mathbf{u}_\beta^\dagger \overline{P}AP\mathbf{u}_\beta & \epsilon^2\mathbf{u}_\beta^\dagger \overline{P}APA\overline{P}\mathbf{u}_\beta \end{pmatrix}. \tag{3.35}$$

All off-diagonal terms are at least as small as $O(\epsilon)$, but the DHC terms will be

$$\frac{|\mathrm{Im}(\mathbf{u}_\alpha^\dagger PA\overline{P}\mathbf{u}_\beta)|}{||P\mathbf{u}_\alpha||\,||PA\overline{P}\mathbf{u}_\beta||} \quad \text{and} \quad \frac{|\mathrm{Re}(\mathbf{u}_\alpha^\dagger \overline{P}APA\overline{P}\mathbf{u}_\beta)|}{||PA\overline{P}\mathbf{u}_\alpha||\,||PA\overline{P}\mathbf{u}_\beta||}, \tag{3.36}$$

which are $O(1)$, and not necessarily small. These terms (3.36) are *discontinuous* as ϵ varies, because for $\epsilon = 0$ they are ill defined since they correspond to $0/0$, and for $\epsilon > 0$ they can take any value between 0 and 1. Since these terms are the overlap of two unit vectors in a $\mathrm{Rank}(P)$ dimensional space, the primary determining factor is the rank of P.

To proceed further one must make assumptions about A. If one wants to consider a random perturbation a natural requirement is that there is no preferred direction, hence A is drawn from a distribution invariant under the unitary group. Random Hermitian matrices of this form have been much studied [31]. Approximate expectations can then be calculated for the terms (3.36) and are $[\mathrm{Rank}(P)]^{-1/2}$ which will always be much larger than $1/(2n)$. Since *every* off-diagonal element must satisfy the DHC, only in exceptional cases will a slightly perturbed set of projectors lead to a set that satisfy the DHC. This effect occurs because the histories that are null in the limit of exact consistency (as $\overline{P}P' \to 0$) now have a finite, though very small, probability.

This is a useful feature as these almost-null histories are uninteresting. Only if there is some special relation between the states, the projectors and the perturbation will the new set be consistent, in which case it is adding information.

The exception to the above occurs if a *single*, binary, branch-dependent projection set is used, and *all* the histories lie in the null space of one of the projectors. Use the projection

set $\{P', \overline{P'}\}$ on the history $\mathbf{u}_1$ when $\overline{P}\mathbf{u}_\alpha = 0$ for all α. Then the real part of the new decoherence matrix to leading order in ϵ contains blocks like

$$\mathrm{Re}\begin{pmatrix} \mathbf{u}_1^\dagger P \mathbf{u}_1 & 0 & -\epsilon^2 \mathbf{u}_1^\dagger A \overline{P} A \mathbf{u}_\alpha \\ 0 & \epsilon^2 \mathbf{u}_1^\dagger A \overline{P} A \mathbf{u}_1 & \epsilon^2 \mathbf{u}_1^\dagger A \overline{P} A \mathbf{u}_\alpha \\ -\epsilon^2 \mathbf{u}_\alpha^\dagger A \overline{P} A \mathbf{u}_1 & \epsilon^2 \mathbf{u}_\alpha^\dagger A \overline{P} A \mathbf{u}_\alpha & \mathbf{u}_\alpha^\dagger \mathbf{u}_\alpha \end{pmatrix}. \tag{3.37}$$

All the off-diagonal terms are $O(\epsilon^2)$ and all the diagonal terms are $O(1)$, except for one which is $O(\epsilon^2)$, so this set satisfies the DHC to $O(\epsilon)$. This is a very special case and is unlikely to occur in real examples.

D. Other Properties of the Dowker-Halliwell Criterion

In standard Quantum Mechanics the probability of observing a system in state $|\phi\rangle$ when it is in state $|\psi\rangle$ is $|\langle\phi|\psi\rangle|^2/(\langle\phi|\phi\rangle\,\langle\psi|\psi\rangle)$. If we take this as a measure of distinguishability then the set of history states, $\{\mathbf{u}_\alpha\}$, are distinguishable to order ϵ^2 only if

$$\frac{|{\mathbf{u}_\alpha}^\dagger \mathbf{u}_\beta|^2}{||\mathbf{u}_\alpha||^2 ||\mathbf{u}_\beta||^2} \leq \epsilon^2 \quad \forall \alpha \neq \beta. \tag{3.38}$$

But this is equivalent to the medium DHC (3.1) since

$$\frac{|{\mathbf{u}_\alpha}^\dagger \mathbf{u}_\beta|}{||\mathbf{u}_\alpha||\,||\mathbf{u}_\beta||} = \frac{|D_{\alpha\beta}|}{(D_{\alpha\alpha} D_{\beta\beta})^{1/2}}. \tag{3.39}$$

Histories which only satisfy the weak consistency criterion (1.4) need not be distinguishable since a pair of histories may only differ by a factor of i and would be regarded as equivalent in conventional quantum mechanics. This is one of the few differences between the medium (3.5) and the standard (3.1) DHC.

Outside of quantum cosmology one usually discusses conditional probabilities: one regards the past history of the universe as definite and estimates probabilities for the future from it. One does this in consistent histories by forming the *current density matrix* ρ_c. Let $\{C_\alpha\}$ be a complete set of class-operators, each of which can be divided into the the past and the future, $C_\alpha = C^f_{\alpha_f} C^p_{\alpha_p}$. Then the probability of history α_f occuring given α_p has occurred is

$$P(\alpha_f|\alpha_p) = \frac{P(\alpha_f \,\&\, \alpha_p)}{P(\alpha_p)}, \tag{3.40}$$

$$= \frac{\mathrm{Tr}(C^f_{\alpha_f} C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p} C^{f\dagger}_{\alpha_f})}{\mathrm{Tr}(C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p})}, \tag{3.41}$$

$$= \mathrm{Tr}(C^f_{\alpha_f} \rho_c C^{f\dagger}_{\alpha_f}), \tag{3.42}$$

where $\rho_c = C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p} / \mathrm{Tr}(C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p})$. Equation (3.42) shows that all future probabilities can be expressed in terms of ρ_c. The DHC in terms of ρ_c is

$$\frac{\mathrm{Tr}(C^f_{\gamma_f} \rho_c C^{f\dagger}_{\beta_f})}{[\mathrm{Tr}(C^f_{\gamma_f} \rho_c C^{f\dagger}_{\gamma_f}) \mathrm{Tr}(C^f_{\beta_f} \rho_c C^{f\dagger}_{\beta_f})]^{1/2}} \leq \epsilon, \quad \forall \gamma_f \neq \beta_f. \tag{3.43}$$

This is the same as the DHC applied to the complete histories, given the past,

$$\frac{\mathrm{Tr}(C^f_{\gamma_f} C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p} C^{f\dagger}_{\beta_f})}{[\mathrm{Tr}(C^f_{\gamma_f} C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p} C^{f\dagger}_{\gamma_f}) \mathrm{Tr}(C^f_{\beta_f} C^p_{\alpha_p} \rho C^{p\dagger}_{\alpha_p} C^{f\dagger}_{\beta_f})]^{1/2}} \leq \epsilon, \tag{3.44}$$

for all $\gamma_f \neq \beta_f$. This is a property not possessed by the usual criterion (2.4) or any other based on absolute probabilities, such as one that only bounds the MPV. This is an important property since any non-trivial[9] branch of a consistent set of histories (when regarded as a set of histories in its own right) must also be consistent and one would like a criterion for approximate consistency that reflects this.

Experiments in quantum mechanics are usually carried out many times, and the relative frequencies of the outcomes checked with their probabilities predicted by quantum mechanics. Consider the situation where an experiment is carried out at m times $\{t_i\}$ with probabilities $\{p_i\}$. Let P^i be the projector corresponding to the experiment being performed at t_i and let $\{C^i_\alpha\} = \{U(-t_i) C_\alpha U(t_i)\}$ be the set of n class-operators corresponding to the different outcomes of the experiment when it is started at time t_i. For simplicity assume that the probability of an experiment being performed and its results are independent of other events. This implies $[P^i, P^j] = 0$, $[P^i, C^j_\alpha] = 0$ and $[C^i_\alpha, C^j_\alpha] = 0$ so $p_i = \langle \phi | P^i | \phi \rangle$. There are $(1+n)^m$ class-operators and they are of the form

---

[9]i.e. a branch whose probability is non-zero.

$$\overline{P^{i_k}}\ldots\overline{P^{i_1}}\,P^{j_{m-k}}\ldots P^{j_1}\,C^{j_{m-k}}_{\alpha_{m-k}}\ldots C^{j_1}_{\alpha_1}, \tag{3.45}$$

corresponding to the experiment being performed at times $t_{j_1}\ldots t_{j_{m-k}}$ and not at times $t_{i_1}\ldots t_{i_k}$ with results $\alpha_1\ldots\alpha_{m-k}$. Because of the commutation relations the only non-zero off-diagonal-elements of the decoherence matrix contain factors like

$$p_i\,\mathrm{Re}(\langle\psi|C^\dagger_\beta C_\alpha|\psi\rangle), \tag{3.46}$$

where $|\psi\rangle$ is the initial state in which the experiment is prepared identically each time. When the environment and time between experiments are large the P_i will commute and this justifies the usual arguments where the consistency of the experiment alone is considered rather than the consistency of the entire run of experiments.

This is a particular case of the result that an inconsistent set cannot extend a non-trivial branch of a set of histories without destroying its consistency. A sensible criterion for approximate consistency should also have this property. By choosing the p_i small enough the off-diagonal elements (3.46) can be made arbitrarily small, thus any criterion for approximate consistency which uses absolute probabilities will regard the set as consistent, however inconsistent the experiment itself may be.

An important feature of the DHC is that it has no such disadvantage, as the p_i's will cancel and the approximate consistency conditions will be

$$\frac{|\mathrm{Re}(\langle\psi|C^\dagger_\beta C_\alpha|\psi\rangle)|}{||C_\alpha|\psi\rangle||\,||C_\beta|\psi\rangle||}\le\epsilon. \tag{3.47}$$

IV. CONCLUSIONS

A set of histories is approximately consistent to order δ, only if its MPV is less than δ. The often-used criterion

$$\mathrm{Re}(D_{\alpha\beta})\le\epsilon(\delta),\quad\forall\alpha\ne\beta \tag{4.1}$$

is not sufficient for any $\epsilon(\delta) > 0$, since there are sets of histories satisfying (4.1) with arbitrarily large MPV. The criterion (4.1) can only be used if $\epsilon(\delta) = O(1/n^2)$, where n is the number of histories. The Dowker-Halliwell criterion has no such disadvantage.

If

$$\mathrm{Re}(D_{\alpha\beta}) \leq \frac{\delta}{2d}\,(D_{\alpha\alpha}D_{\beta\beta})^{1/2} \quad \forall \alpha \neq \beta,\, \delta < 1, \tag{4.2}$$

then the MPV is less than $\delta + O(\delta^2)$. This is the paper's main result. If the medium DHC holds,

$$|D_{\alpha\beta}| \leq \frac{\delta}{d}\,(D_{\alpha\alpha}D_{\beta\beta})^{1/2} \quad \forall \alpha \neq \beta,\, \delta < 1, \tag{4.3}$$

then the MPV is also bounded by δ. For histories satisfying either criterion, if only homogenous class-operators are used then the upper bound on the MPV is strengthened to $\delta/2$. The bounds are also optimal in the sense that they are can be achieved (to within a small factor) in any finite dimensional Hilbert space. Any improved bound must use the global structure of the decoherence matrix.

The DHC is particularly suitable for computer models in which a set of histories is built up by repeated projections. If each history satisfies (4.2) as it is added, then the whole set will be consistent to order δ and there will be no more than $2d$ histories.

The DHC also leads to a simple, geometrical picture of consistency: the path-projected states can be regarded as pairs of points on the surface of a hyper-sphere, all separated by at least $\cos^{-1}\epsilon$ degrees. This approach can be used to prove that ϵ in the DHC must be chosen according to the dimension of the Hilbert space. Ideally one would like a criterion for approximate consistency that implied the existence of an exactly consistent set corresponding to physical events that only differed to order ϵ. The DHC seems well adapted to defining proximity to an exactly consistent set and may be useful in constructing a proof that such a set exists.

This bound (3.29) shows that the number of histories satisfying the DHC can increase exponentially with d if ϵ is constant. So for constant $\epsilon > 0$ by choosing a large enough

Hilbert space the MPV can be arbitrarily large, therefore ϵ must be chosen according to the dimension of the Hilbert space.

If a set is not exactly consistent then it cannot be a subset of an exactly consistent set (unless the branch is trivial.) The same is true for approximate consistency when it is defined by the DHC. However, this is not true however for any criterion which depends solely on MPV. It is a particularly useful property when discussing conditional probabilities.

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APPENDIX A: SPHERE-PACKING BOUNDS

1. Upper Bounds Using Zonal Spherical Harmonic Polynomials

Various authors [32,33] have constructed upper bounds for M by using the properties of zonal spherical harmonic polynomials, which for many spaces are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The bounds

$$M(\mathbf{S}^{d-1},\, |\mathbf{u}^T\mathbf{v}| \le \epsilon) = N((d-3)/2, -1/2, 2\epsilon^2 - 1), \tag{A1}$$

for $d \ge 3$, and

$$M(\mathbf{CS}^{d-1},\, |\mathbf{u}^\dagger\mathbf{v}| \le \epsilon) = N(d-2, 0, 2\epsilon^2 - 1), \tag{A2}$$

for $d \ge 2$, have been proved by Kabatyanski et al. [32] and (A1) also by Delsarte et al. [33]. Here $N(\alpha,\beta,s)$ is defined as the solution to the following optimisation problem.

Consider s as a given number $-1 \le s < 1$. Let $\mathcal{R}(\alpha,\beta,s)$ be the set of polynomials of degree at most k with the following properties:

$$f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(t),$$

$$f_i \geq 0, \quad i = 0, 1, \ldots, k, \quad \text{and} \quad f_0 > 0,$$

$$f(t) \leq 0 \quad \text{for} \quad -1 \leq t \leq s.$$

Then

$$N(\alpha, \beta, s) = \inf_{f(t) \in \mathcal{R}(\alpha,\beta,s)} f(1)/f_0.$$

This can be converted to a linear program by defining

$$\tilde{P}_i^{(\alpha,\beta)}(t) = P_i^{(\alpha,\beta)}(t)/P_i^{(\alpha,\beta)}(1).$$

Then $N(\alpha, \beta, s) = 1 + \sum_{i=1}^{k} f_i$, where $\sum_{i=1}^{k} f_i$ is minimised subject to $f_i \geq 0$ and $\sum_{i=1}^{k} f_i \tilde{P}_i^{(\alpha,\beta)}(t) \leq -1$, for $-1 \leq t \leq s$. This formulation is discussed in Conway and Sloane [34], but no exact solutions are known. However, any $f(t)$ satisfying the constraints does provide a bound, though it may not be optimal. I show in appendix B 1 that

$$\tilde{P}_n^{(\alpha,-1/2)}(x) > \tilde{P}_1^{(\alpha,-1/2)}(x),$$

if $-1 < x < -(2\alpha + 3)(2\alpha + 5)^{-1}$, $n > 1$ and $\alpha \geq 1$. So if s is less than $-(2\alpha + 3)/(2\alpha + 5)$ then $\tilde{P}_1^{(\alpha,-1/2)}(t)$ is more negative than any other of the $\tilde{P}_i^{(\alpha,-1/2)}(t)$, and since $\tilde{P}_1^{(\alpha,-1/2)}(t)$ is increasing the solution is

$$f_i = 0, \quad i = 2, 3, \ldots, \quad f_1 = -1/\tilde{P}_1^{(\alpha,-1/2)}(s) \quad \forall k.$$

So the optimal bound using zonal spherical harmonics is

$$\begin{aligned} M(\mathbf{CS}^{d-1}, \mathrm{Re}(\mathbf{u}^\dagger \mathbf{v}) \leq \epsilon) &= M(\mathbf{S}^{2d-1}, |\mathbf{u}^T \mathbf{v}| \leq \epsilon) \\ &\leq N(d - 3/2, -1/2, 2\epsilon^2 - 1), \\ &= 1 - 1/\tilde{P}_1^{(d-3/2,-1/2)}(2\epsilon^2 - 1) \\ &= \frac{2d(1 - \epsilon^2)}{1 - 2d\epsilon^2}, \end{aligned} \tag{A3}$$

if $\epsilon^2 \leq 1/(2d + 2)$ and $d \geq 3$. I prove a similar inequality in appendix B 2 for $\alpha = 0$. So for the medium DHC

$$M(\mathbf{CS}^{d-1}, |\mathbf{u}^\dagger \mathbf{v}| \leq \epsilon) \leq N(d-2, 0, 2\epsilon^2 - 1),$$
$$= \frac{d(1-\epsilon^2)}{1 - d\epsilon^2}, \tag{A4}$$

if $\epsilon^2 \leq 1/(d+1)$ and $d \geq 2$.

2. Shannon's Lower Bound

In a pioneering paper [29] Shannon proved

Theorem 1

$$M(\mathbf{S}^{d-1}, |\mathbf{u}^T \mathbf{v}| \leq \cos\theta) \geq \sin^{1-d}\theta. \tag{A5}$$

Let

$$S_d(r) = dr^{d-1}\pi^{d/2}/\Gamma[(d+2)/2] \tag{A6}$$

be the surface area of a sphere in Euclidean d-space of radius r, and let $A_d(r, \theta)$ be the area of a d-dimensional spherical cap cut from a sphere of radius r with half angle θ. It is not hard to show that

$$A_d(r, \theta) = \frac{(d-1)r^{d-1}\pi^{d-1/2}}{\Gamma[(d+2)/2]} \int_0^\theta \sin^{d-2}\phi \, d\phi. \tag{A7}$$

Consider the largest possible set of rays through the origin intersecting a sphere at points points $\mathbf{u} \in \mathbf{S}^{d-1}$. About each point $\mathbf{u}$, consider the spherical cap of all points on the sphere within θ degrees. Now, the set of all such caps about each point $\mathbf{u}$ must cover the entire surface of the sphere, otherwise we could add a new ray passing through the uncovered areas. Since the area of each cap is $A_d(r, \theta)$, we have

$$2\, A_d(r, \theta)\, M(\mathbf{S}^{d-1}, |\mathbf{u}^T \mathbf{v}| \leq \cos\theta) \;\geq\; S_d(r). \tag{A8}$$

But a spherical cap, $A_d(r, \theta)$, is contained within a hemisphere of radius $r\sin\theta$, $A_d(r, \theta) \leq 1/2\, S_d(r \sin\theta)$ [10] , so

[10] This is easy to prove by changing variables in the integral to $\sin\phi = \sin\theta \sin\psi$

$$M(\mathbf{S}^{d-1}, |\mathbf{u}^T\mathbf{v}| \le \cos\theta) \ge S_d(r)/S_d(r\sin\theta) = \sin^{1-d}\theta \blacksquare \tag{A9}$$

or

$$M(\mathbf{CS}^{d-1}, \mathrm{Re}(\mathbf{u}^\dagger\mathbf{v}) \le \cos\theta) \ge \sin^{1-2d}\theta. \tag{A10}$$

The straightforward extension of the proof to the complex case does not appear to exist in the literature. It is slightly simpler as it is easy to calculate the integral $A_d(r,\theta)$ exactly.

Theorem 2

$$M(\mathbf{CS}^{d-1}, |\mathbf{u}^\dagger\mathbf{v}| \le \cos\theta) \ge \sin^{2-2d}\theta \tag{A11}$$

The area of a unit sphere in $\mathbf{CS}^{d-1}$ is $S_{2d}(1)$. Let $A_d(1,\theta)$ now be the area of a cap defined by

$$\{\mathbf{u} \in \mathbf{CS}^{d-1} : |u_1|^2 \ge \cos\theta\}. \tag{A12}$$

We can choose coordinates for a vector $\mathbf{u}$ in $\mathbf{CS}^{d-1}$ by defining

$$\begin{aligned}
\mathrm{Re}(u_1) &= \cos\phi_1, \\
\mathrm{Im}(u_1) &= \sin\phi_1\,\cos\phi_2, \\
&\vdots\ \vdots\ \vdots, \\
\mathrm{Re}(u_d) &= \sin\phi_1\,\sin\phi_2\,\sin\phi_3\ldots\sin\phi_{2d-2}\cos\psi, \\
\mathrm{Im}(u_d) &= \sin\phi_1\,\sin\phi_2\,\sin\phi_3\ldots\sin\phi_{2d-2}\sin\psi,
\end{aligned}$$

where $\phi_n \in [0,\pi)$ and $\psi \in [0,2\pi)$.Then, by integrating over $\phi_2, \phi_3, \ldots, \phi_{2d-2}$ and ψ, we get

$$\begin{aligned}
A_d(1,\theta) &= S_{2d-2}(1) \iint\limits_{\cos^2\phi_1+\sin^2\phi_1\cos^2\phi_2 \ge \cos\theta} \sin^{d-2}\phi_1 \sin^{d-3}\phi_1\, d\phi_1\, d\phi_2 \\
&= \frac{\pi S_{2d-2}(1)\sin^{2d-2}\theta}{d-2}.
\end{aligned} \tag{A13}$$

Hence, using Shannon's argument again,

$$
\begin{aligned}
M(\mathbf{CS}^{d-1}, |\mathbf{u}^\dagger \mathbf{v}| \le \cos\theta) &\ge \frac{(d-2)\, S_{2d}(1)}{\pi S_{2d-2}(1) \sin^{2d-2}\theta} \\
&\ge \sin^{2-2d}\theta \; \blacksquare
\end{aligned}
\tag{A14}
$$

Expressed in terms of $\epsilon = \cos\theta$ the bounds are

$$
M(\mathbf{CS}^{d-1}, \mathrm{Re}(\mathbf{u}^\dagger \mathbf{v}) \le \epsilon) \ge (1-\epsilon^2)^{1/2-d},
$$

$$
\text{and} \qquad M(\mathbf{CS}^{d-1}, |\mathbf{u}^\dagger \mathbf{v}| \le \epsilon) \ge (1-\epsilon^2)^{1-d}.
$$

APPENDIX B: JACOBI POLYNOMIALS

I have used trivial properties of the Jacobi polynomials without citation. All of these results can be found in chapter IV of Szegö, [35] which provides an excellent introduction to, and reference source for, the Jacobi polynomials.

1. S^{d-1}, $\beta = -1/2$

In $\mathbf{S}^{d-1}$ the zonal spherical polynomials are $P_n^{(\alpha,-1/2)}(x)$ with $\alpha = (d-3)/2$.

Theorem 3

$$
\tilde{P}_n^{(\alpha,-1/2)}(x) > \tilde{P}_1^{(\alpha,-1/2)}(x) \tag{B1}
$$

for $-1 < x < -(2\alpha+3)(2\alpha+5)^{-1}$, $n > 1$ *and* $\alpha \ge 1$, *where* $\tilde{P}_n^{(\alpha,-1/2)}(x) = P_n^{(\alpha,-1/2)}(x)/P_n^{(\alpha,-1/2)}(1)$.

I begin by considering two special cases, $n = 2$ and $n = 3$. The first four polynomials are:

$$
\begin{aligned}
\tilde{P}_0^{(\alpha,-1/2)}(x) &= 1 \\
\tilde{P}_1^{(\alpha,-1/2)}(x) &= \frac{2\alpha + 1 + (2\alpha+3)x}{4(\alpha+1)} \\
\tilde{P}_2^{(\alpha,-1/2)}(x) &= \frac{4\alpha^2 - 13 + 2(2\alpha+1)(2\alpha+5)x + (2\alpha+5)(2\alpha+7)x^2}{16(\alpha+1)(\alpha+2)}
\end{aligned}
$$

$$\tilde{P}_3^{(\alpha,-1/2)}(x) = \frac{(2\alpha+1)(4\alpha^2-8\alpha-57)+3(2\alpha+7)(4\alpha^2-21)x}{\frac{+3(2\alpha+1)(2\alpha+7)(2\alpha+9)x^2}{\frac{+(2\alpha+7)(2\alpha+9)(2\alpha+11)x^3}{64(\alpha+1)(\alpha+2)(\alpha+3)}}}.$$

So

$$\tilde{P}_2^{(\alpha,-1/2)}(x) - \tilde{P}_1^{(\alpha,-1/2)}(x) = \frac{-(2\alpha+7)(1-x)[2\alpha+3+(2\alpha+5)x]}{16\,(\alpha+1)(\alpha+2)} \tag{B2}$$

$$\tilde{P}_3^{(\alpha,-1/2)}(x) - \tilde{P}_1^{(\alpha,-1/2)}(x) =$$
$$\frac{-(2\alpha+9)(1-x)[(2\alpha+1)(6\alpha+17)+2\,(2\alpha+7)(4\alpha+7)x}{\frac{+(2\alpha+7)(2\alpha+11)x^2]}{64\,(\alpha+1)(\alpha+2)(\alpha+3)}}. \tag{B3, B4}$$

Equation (B2) is positive for $x < -(2\alpha+3)(2\alpha+5)^{-1}$ (hence the range chosen for (B1).) Equation (B4) is positive where the quadratic factor

$$(2\alpha+1)(6\alpha+17)+2(2\alpha+7)(4\alpha+7)x + (2\alpha+7)(2\alpha+11)x^2 \tag{B5}$$

is negative. Since (B5) is positive for large $|x|$ if it is negative at any two points it will be negative in between. At $x=-1$ it is $-4\,(2\alpha+1)$, and at $x=-(2\alpha+3)(2\alpha+5)^{-1}$ it is $-16\,(\alpha+2)(2\alpha+11)(5+2\alpha)^{-2}$, which is negative for $\alpha > -2$. So the inequality (B1) holds for $n=2$ and $n=3$.

For $n>3$ the inequality is easily proved, by bounding the solutions of the Jacobi differential equation,

$$(1-x^2)y''(x)+[\beta-\alpha-(\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x)=0, \tag{B6}$$

where $y(x)=P_n^{(\alpha,\beta)}(x)$. Define $w(s)=(1-s^2)^\alpha y(2s^2-1)$, $s\in[0,1]$. Substituting $\beta=-1/2$ into equation (B6) it becomes

$$\left[\frac{w'(s)}{(1-s^2)^{\alpha-1}}\right]' + \frac{2(\alpha+n)(1+2n)w(s)}{(1-s^2)^{\alpha}} = 0, \tag{B7}$$

which is of the form

$$[k(s)w'(s)]' + \phi(s)w(s) = 0$$

with $k(s)$ and $\phi(s)$ positive, and $k(s)\phi(s)$ increasing, if α and n are positive. These are the necessary conditions for the Sonine-Pölya theorem (appendix C), which states that the local maxima of $|w(s)|$ will be decreasing. From its definition $|w(s)|$ has a local maximum at $s = 0$, since $w(0)w''(0) < 0$, and a local minimum at $s = 1$, since $w(0) = 0$. $w(s)$ is continuous so it is bounded by its local maxima, hence $|w(s)| \le |w(0)|$, for $s \in [0, 1]$. In the original variables this is

$$\left(\frac{1-x}{2}\right)^{\alpha} \left|P_n^{(\alpha,-1/2)}(x)\right| \le \left|P_n^{(\alpha,-1/2)}(-1)\right|. \tag{B8}$$

Substituting in the values of $P_n^{(\alpha,-1/2)}(-1)$ and $P_n^{(\alpha,-1/2)}(1)$ this becomes[11]

$$|\tilde{P}_n^{(\alpha,-1/2)}(x)| \le \frac{(1/2)_n}{(\alpha+1)_n}\left(\frac{2}{1-x}\right)^{\alpha}, \tag{B9}$$

for $-1 \le x \le 1$. The right hand side is decreasing with n if $\alpha > -1/2$. So for $n \ge 4$

$$|\tilde{P}_n^{(\alpha,-1/2)}(x)| < \frac{105/16}{(\alpha+1)_4}\left(\frac{2}{1-x}\right)^{\alpha}. \tag{B10}$$

This is increasing with x so achieves its maximum at $x = -(2\alpha+3)/(2\alpha+5)$. Thus

$$|\tilde{P}_n^{(\alpha,-1/2)}(x)| \le \frac{105/16}{(\alpha+1)_4}\left(\frac{2\alpha+5}{2\alpha+4}\right)^{\alpha}. \tag{B11}$$

For $\alpha \ge 1$ this is strictly bounded by

$$\frac{1}{(\alpha+1)(2\alpha+5)} = \left|\tilde{P}_1^{(\alpha,-1/2)}\left(-\frac{2\alpha+3}{2\alpha+5}\right)\right|, \tag{B12}$$

and since it is decreasing and $x \le -(2\alpha+3)(2\alpha+5)^{-1}$

[11] The Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\dots(a+n-1)$.

$$|\tilde{P}_n^{(\alpha,-1/2)}(x)| < |\tilde{P}_1^{(\alpha,-1/2)}(x)|. \tag{B13}$$

But $\tilde{P}_1^{(\alpha,-1/2)}(x)$ is negative on the range of x so

$$\tilde{P}_n^{(\alpha,-1/2)}(x) > \tilde{P}_1^{(\alpha,-1/2)}(x)\ \blacksquare \tag{B14}$$

2. CS^{d-1}, $\beta = 0$

In $\mathbf{CS}^{d-1}$ the zonal spherical polynomials are $P_n^{(\alpha,0)}(x)$, where $\alpha = d-2$, and a similar theorem exists.

Theorem 4

$$\tilde{P}_n^{(\alpha,0)}(x) > \tilde{P}_1^{(\alpha,0)}(x), \tag{B15}$$

for $-1 < x < -(\alpha+1)(\alpha+3)^{-1}$, $n > 1$ *and* $\alpha \geq 2$, *where* $\tilde{P}_n^{(\alpha,0)}(x) = P_n^{(\alpha,0)}(x)/P_n^{(\alpha,0)}(1)$.

I begin by considering two special cases, $n = 2$ and $n = 3$. The first four polynomials are

$$\begin{aligned}
\tilde{P}_0^{(\alpha,0)}(x) &= 1\\
\tilde{P}_1^{(\alpha,0)}(x) &= \frac{\alpha+(\alpha+2)x}{2(\alpha+1)}\\
\tilde{P}_2^{(\alpha,0)}(x) &= \frac{\alpha^2-\alpha-4+2\alpha(\alpha+3)x+(\alpha+3)(\alpha+4)x^2}{4(\alpha+1)(\alpha+2)}\\
\tilde{P}_3^{(\alpha,0)}(x) &= \frac{\alpha(\alpha^2-3\alpha-16)+3(\alpha-3)(\alpha+2)(\alpha+4)x+3\alpha(\alpha+4)(\alpha+5)x^2}{}\\
&\quad \frac{+(\alpha+4)(\alpha+5)(\alpha+6)x^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}
\end{aligned}$$

So

$$\tilde{P}_2^{(\alpha,0)}(x) - \tilde{P}_1^{(\alpha,0)}(x) = -\frac{(\alpha+4)(1-x)(\alpha+1+(3+\alpha)x)}{4(\alpha+1)(\alpha+2)} \tag{B16}$$

$$\tilde{P}_3^{(\alpha,0)}(x) - \tilde{P}_1^{(\alpha,0)}(x) = -\frac{(\alpha+5)(1-x)[\alpha(3\alpha+8)+2(\alpha+4)(2\alpha+3)x+(\alpha+4)(\alpha+6)x^2}{8(\alpha+1)(\alpha+2)(\alpha+3)} \tag{B17}$$

Equation (B16) is positive for $x < -(\alpha+1)(\alpha+3)^{-1}$ (hence the range chosen for (B15).) Equation (B17) is positive when the quadratic factor

$$\alpha(3\alpha+8)+2(\alpha+4)(2\alpha+3)x+(\alpha+4)(\alpha+6)x^2, \tag{B18}$$

is negative. At $x=-1$ equation (B18) equals -4α, and at $x=-(\alpha+1)/(\alpha+3)$ it equals $-4(\alpha+2)(\alpha+6)(\alpha+3)^{-2}$ both of which are negative for $\alpha>0$. Therefore inequality (B15) holds for $n=2$ and $n=3$.

The method in the previous appendix cannot be used unless $\beta=\pm 1/2$, since the differential equation has a singular point at $x=-1$. There is a simple method[12] for the special case of $\beta=0$, based on a result from Szegö (7.21.2) [35]

$$[(1-x)/2]^{\alpha/2+1/4}\left|P_n^{(\alpha,0)}(x)\right| \leq 1, \tag{B19}$$

when $-1\leq x\leq 1$ and $\alpha\geq -1/2$. Substituting $P_n^{(\alpha,0)}(1)=(\alpha+1)_n/n!$ into (B19) it becomes

$$\left|\tilde{P}_n^{(\alpha,0)}(x)\right| \leq \frac{n!}{(\alpha+1)_n}\left(\frac{2}{1-x}\right)^{(\alpha/2+1/4)}. \tag{B20}$$

For $\alpha>0$ the right hand side is decreasing with n, so for $n\geq 4$

$$\left|\tilde{P}_n^{(\alpha,0)}(x)\right| \leq \frac{4!}{(\alpha+1)_4}\left(\frac{2}{1-x}\right)^{(\alpha/2+1/4)}. \tag{B21}$$

This is decreasing with x so achieves its maximum at $x\leq -(\alpha+1)(\alpha+3)^{-1}$. Thus

$$\left|\tilde{P}_n^{(\alpha,0)}(x)\right| \leq \frac{4!}{(\alpha+1)_4}\left(\frac{\alpha+2}{\alpha+3}\right)^{(\alpha/2+1/4)}. \tag{B22}$$

For $\alpha\geq 2$ this is strictly bounded by

$$\frac{1}{(\alpha+1)(\alpha+3)} = \left|\tilde{P}_1^{(\alpha,0)}\left(-\frac{\alpha+1}{\alpha+3}\right)\right|. \tag{B23}$$

Since $|\tilde{P}_1^{\alpha}(x)|$ is monotonic increasing

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[12] Szegö proves that for polynomials $p(s)$ orthogonal with weight function $w(s)$, that if $w(s)$ is non-decreasing then $[w(s)]^{1/2}|p(s)|$ is non-decreasing also. The weight measure over which $P^{(\alpha,0)}$ are orthogonal is $(1-x)^{\alpha}$dx. After changing variable to $x=2s^2-1$ the new measure is proportional to $s^{(2\alpha+1)}$ds, which is non-decreasing.

$$\left|\tilde{P}_n^{(\alpha,0)}(x)\right| < \left|\tilde{P}_1^{(\alpha,0)}(x)\right|. \tag{B24}$$

But $\tilde{P}_1^{\alpha}(x)$ is negative on the range so

$$\tilde{P}_n^{(\alpha,0)}(x) > \tilde{P}_1^{(\alpha,0)}(x) \blacksquare$$

APPENDIX C: SONINE-PÖLYA THEOREM

This standard theorem is referred to in [35, 7.31.2].

Theorem 5 *Let $y(x)$ be a solution of the differential equation*

$$[k(x)y'(x)]' + \phi(x)y(x) = 0.$$

If $k(x)$ and $\phi(x)$ are positive, and $k(x)\phi(x)$ is increasing (decreasing) and its derivative exists, then the local maxima of $|y(x)|$ are decreasing (increasing).

Let

$$f(x) = [y(x)]^2 + [k(x)y'(x)]^2[k(x)\phi(x)]^{-1}$$

then $f(x) = [y(x)]^2$ if $y'(x) = 0$, and

$$\begin{aligned} f' &= 2y'\left\{y + \frac{[ky']'}{k\phi} - \frac{[k\phi]'y'}{2\phi^2}\right\} \\ &= -\frac{y'^2[k\phi]'}{\phi^2}. \end{aligned}$$

So $\mathrm{sgn} f'(x) = -\mathrm{sgn}[k(x)\phi(x)]'$ $\blacksquare$

APPENDIX D: AN EXAMPLE OF LARGE PROBABILITY VIOLATION

Consider the $2n$ vectors

$$(\mathbf{u}_i)_j = \frac{a\delta_{ij} - 1}{\sqrt{a^2 - 2a + n}} \in \mathbf{H}_1, \tag{D1}$$

$$(\mathbf{v}_i)_j = \frac{b\delta_{ij} + 1}{\sqrt{b^2 + 2b + n}} \in \mathbf{H}_2, \tag{D2}$$

where

$$a = \frac{1+\epsilon+\sqrt{(1+\epsilon)(1+\epsilon-n\epsilon)}}{\epsilon}, \tag{D3}$$

$$b = \frac{1-\epsilon+\sqrt{(1-\epsilon)(1-\epsilon+n\epsilon)}}{\epsilon} \tag{D4}$$

and $\epsilon \leq 1/(n-1)$. Then

$$\mathbf{u}_i{}^\dagger \mathbf{u}_j = \delta_{ij}(1+\epsilon)-\epsilon \quad \text{and} \quad \mathbf{v}_i{}^\dagger \mathbf{v}_j = \delta_{ij}(1-\epsilon)+\epsilon. \tag{D5}$$

Define

$$\mathbf{w}_i = (\mathbf{u}_i \oplus \mathbf{u}_i)/\sqrt{2} \in \mathbf{H}_1 \oplus \mathbf{H}_2, \tag{D6}$$

so

$$\mathbf{w}_i^\dagger \mathbf{w}_j = \delta_{ij}. \tag{D7}$$

Let the initial state be

$$\psi = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{w}_i \in \mathbf{H}_1 \oplus \mathbf{H}_2. \tag{D8}$$

Then use $\{\mathbf{w}_i \mathbf{w}_i^\dagger\}$ as a set of projectors to get the history states

$$\left\{\mathbf{w}_1/\sqrt{n}, \ldots, \mathbf{w}_n/\sqrt{n}\right\} \tag{D9}$$

Then make a projection onto $\mathbf{H}_1$ and $\mathbf{H}_2$ to get the history states

$$\left\{\mathbf{u}_1/\sqrt{2n}, \ldots, \mathbf{u}_n/\sqrt{2n}, \mathbf{v}_1/\sqrt{2n}, \ldots, \mathbf{v}_n/\sqrt{2n}\right\}. \tag{D10}$$

The decoherence matrix can be written

$$\frac{1}{2n} \begin{pmatrix} 1 & -\epsilon & \ldots & -\epsilon & 0 & \ldots & \ldots & 0 \\ -\epsilon & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\epsilon & \vdots & \ddots & \ddots & \vdots \\ -\epsilon & \ldots & -\epsilon & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & 1 & \epsilon & \ldots & \epsilon \\ \vdots & \ddots & \ddots & \vdots & \epsilon & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \epsilon \\ 0 & \ldots & \ldots & 0 & \epsilon & \ldots & \epsilon & 1 \end{pmatrix}. \tag{D11}$$

The MPV for this set is $|-n(n-1)\epsilon/(2n)| = (n-1)\epsilon/2 \approx d\epsilon/4$. It is achieved by coarse-graining all the $\mathbf{u}_i$'s (or $\mathbf{v}_i$'s) together.

APPENDIX E: QUANTUM ZENO EFFECT

The Quantum Zeno effect is often discussed in the interpretation of quantum mechanics, but has had no quantitative analysis in the consistent histories formalism.

Consider a two dimensional Hilbert space. Define the vectors

$$\mathbf{u}_+^n = \begin{pmatrix} \cos(n\epsilon) \\ \sin(n\epsilon) \end{pmatrix}, \mathbf{u}_-^n = \begin{pmatrix} -\sin(n\epsilon) \\ \cos(n\epsilon) \end{pmatrix}, \tag{E1}$$

and the projectors

$$P_+^n = \mathbf{u}_+^n \mathbf{u}_+^{n\,\dagger}, P_-^n = \mathbf{u}_-^n \mathbf{u}_-^{n\,\dagger}. \tag{E2}$$

For any n, P_+^n and P_-^n are a complete set of projectors. Consider the set of histories formed by using strings of these projectors on the initial state $\mathbf{u}_+^0$.

$$C_\alpha = P_{\alpha_n}^n \dots P_{\alpha_1}^1. \tag{E3}$$

The histories α are string of n pluses or minuses.

Define $|\alpha|$ to be the number of transitions from plus to minus or vice versa in the string $\{\alpha_1, \dots, \alpha_n, +\}$. Then

$$C_\alpha \mathbf{u}_+^0 = \mathbf{u}_{\alpha_n}^n (-1)^{\lfloor \frac{|\alpha|+1}{2} \rfloor} \cos^{n-|\alpha|}\epsilon \sin^{|\alpha|}\epsilon, \tag{E4}$$

and there will be $\binom{n}{|\alpha|}$ identical histories states. The non-zero decoherence matrix elements are those with $|\alpha| = |\beta|$ mod 2 and are

$$D_{\alpha\beta} = (-1)^{\lfloor \frac{|\alpha|+1}{2} \rfloor} (-1)^{\lfloor \frac{|\beta|+1}{2} \rfloor} \cos^{2n-|\alpha|-|\beta|}\epsilon \sin^{|\alpha|+|\beta|}\epsilon, \tag{E5}$$

Because of the simple form of (E5) all of the following calculations can be done exactly, but for simplicity I shall let $\epsilon = \theta/n$ and work to leading order in $1/n$. The largest probability

violation for this decoherence matrix will be achieved by coarse-graining together all the histories with a positive sign into one history and all those with a negative sign into another. Let X denote the histories $|\alpha| = 0, 3 \bmod 4$ and Y the histories $|\alpha| = 1, 2 \bmod 4$. Then the probability violations for these sets are,

$$\sum_{\alpha\neq\beta\in X} D_{\alpha\beta} = 1/2\cosh^2\theta + 1/2\cos\theta\cosh\theta \\ - 1/2\sin\theta\sinh\theta - 1 + O(1/n)$$

for X and

$$\sum_{\alpha\neq\beta\in Y} D_{\alpha\beta} = 1/2\cosh^2\theta - 1/2\cos\theta\cosh\theta \\ + 1/2\sin\theta\sinh\theta + O(1/n)$$

for Y. The off-diagonal elements in the decoherence matrix (E5) are all less than θ^2/n^2 yet the MPV is order $\exp(2\theta)$, so by choosing $n \gg \theta \gg 1$ the off-diagonal elements can be made arbitrarily small whilst the MPV is arbitrarily large. This proves the following theorem.

Theorem 6 *For all Hilbert spaces of dimension* ≥ 2, $\epsilon > 0$ *and* $x > 0$ *there exist finite sets of histories such that*

$$|D_{\alpha\beta}| \leq \epsilon, \quad \forall\alpha \neq \beta, \tag{E6}$$

and with MPV $> x$.

Now suppose the limit $n \to \infty$ is taken. Then all the elements of the decoherence matrix (E5) are zero except for $D_{\alpha\alpha} = 1$, $\alpha = \{+\cdots+\}$. A naive argument would be to say that since all the off-diagonal elements are zero the set is consistent, but this is false. The set is pathologically inconsistent.

This shows that care must be taken with infinite sets of histories. It is incorrect to take the limit of a set of histories and then apply consistency criteria. Instead the order must be reversed and the limit of the criteria taken. This does not always seem to have been recognised in the literature. For instance Halliwell [36] states : "In particular, it [$|D_{\alpha\beta}| \leq (D_{\alpha\alpha}D_{\beta\beta})^{1/2}$] implies that consistency is automatically satisfied if the system has

one history with $D_{\alpha\alpha} = 1$, and $D_{\beta\beta} = 0$ for all other histories." He says this after a similar limit has been taken, and I have shown above that this is not necessarily true.

The DHC trivially rejects this family of histories as grossly inconsistent since

$$\frac{D_{\alpha\beta}}{(D_{\alpha\alpha} D_{\beta\beta})^{1/2}} = 1, \quad \text{whenever } |\alpha| = |\beta| \bmod 2. \tag{E7}$$

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